Cantorian Abstraction: A Reconstruction and Defence

In section 1 of his *Beitrage*[i], Cantor explains the notion of cardinal number:

> We will call by the name "power" or "cardinal number" of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements *m* and of the order in which they are given.
>
> We denote the result of this double act of abstraction, the cardinal number or power of M, by
>
> (3) M (double bar).
>
> Since every single element *m*, if we abstract from its nature, becomes a "unit", the cardinal number M (double bar) is a definite aggregate composed of units, and this number has existence in our mind as an intellectual image or projection of the given aggregate M.

Later in section 7 of the same work, he gives an analogous account of order type:

> Every ordered aggregate M has a definite "order type", or more shortly a definite "type", which we will denote by
>
> (2) M (bar).
>
> By this we understand the general concept which results from M if we only abstract from the nature of the elements *m*, and retain the order of precedence among them. Thus the order type M(bar) is itself an ordered aggregate whose elements are units which have the same order of precedence amongst one another as the corresponding elements of M, from which they are derived by abstraction.

Dedekind, in §73 of his *Was sind und was sollen die Zahlen*[ii], gives a somewhat similar account of number-series:

> If in the consideration of a simply infinite system *N* set in order by a transformation _ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation _, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series N*. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind.

What these accounts have in common is a view of abstraction as the process of freeing an object of its peculiar features and a conception of number or order type as the product of such a process. Natural as this approach to abstraction may be, it has not met with much favour among contemporary philosophers. Frege subjected it to scathing criticism in sections 28-44 of the *Grundlagen*; and most philosophers today would agree with Dummett's view of the theory as "misbegotten" and his assessment of Frege's critique as "conclusive"[iii].

I know of only two attempts at defence in the recent philosophical literature. Hallett[iv] finds in Cantor an anticipation of von Neumann's approach to cardinals as sets of ordinals. But by this he means no more than that 'to make sense of Cantor's abstractionism as a *set* theory of number, one is forced to bring in strong ordinal assumptions' (a claim that we shall later see is mistaken). Tait[v], if I understand him correctly, sees Dedekind and Cantor as providing a "reductive" account of number: to talk of numbers in the one case is to talk of the elements of a given omega progression as if they had no properties beyond those of successor; and to talk of cardinal numbers in the other case is to talk as if there were no difference between equinumerous sets. Such an interpretation will perhaps fit Dedekind's words but, as Tait himself concedes[vi], it makes Cantor's conception of cardinal numbers as sets of pure units both ill-motivated and ill-conceived.

The fault of these interpretations lies, to my mind, in their failure to recognize the distinctive ontology that lies behind these accounts. Abstraction, as Cantor and Dedekind conceive it, is ontologically innovative - it leads to objects that are genuinely new. But Hallett and Tait are presumably so reluctant to accept such an ontology themselves that they do their best not to attribute it to others, Hallett by seeing the apparently new objects as objects of a familiar sort and Tait by treating the apparent reference to them as a mere facon de parler. However, as I hope will become clear, it is only by taking the ontology seriously that we can come to a satisfactory view of what these authors had in mind.

Moreover, once this is done, it will emerge that Cantor's and Dedekind's accounts of number and order type are as coherent, interesting and plausible as the more familiar accounts of Russell and Frege and of von Neumann and Zermelo. Whether they are correct is, of course, another matter. The defence provided in this paper is merely of their coherence and relative plausibility, not of their correctness. But I should state for the record that I am inclined to accept the Cantorian account of order type though not the Cantorian or Dedekindian account of number.

In what follows I shall concentrate on the views of Cantor, though it should be clear how what I say will can be modifed to apply to the views of Dedekind. I have not attempted to capture all of the nuances or tensions in Cantor's thought but merely to develop what I take to be its spirit, or central idea. And in developing this idea, I have been guided more by what the idea itself requires than by Cantor's own writings.

The plan of the paper is as follows. I begin by setting out what appear to be decisive objections to the Cantorian account. I then show how these objections can be overcome by making use of the theory of arbitrary objects developed in my book 'Reasoning with Arbitrary Objects'[vii]. The relevant parts of the theory are outlined in section 2; and the application to Cantor's account of number is made in section 3. I show, in section 4, how the approach may be extended to order types and to structure types in general. In the final two sections, I first compare the Cantorian approach to abstraction with the standard approaches of von Neumann and Zermelo, on the one side, and of Russell and Frege, on the other; and I then consider to what extent the Cantorian approach is capable of yielding a structuralist conception of number and order type. In a formal appendix, I briefly indicate how the present theory might be formalized within an extension of ZF.

The present line of investigation ties in with certain other topics which I shall mention but not explore. First, the theory of units developed in the appendix corresponds to a theory of classes in which classes can naturally be taken to be members of sets. This theory is, I believe, of considerable interest in its own right. Second, the Cantorian theory can be extended to provide a more general theory of types - covering not merely the abstract formal types of mathematics but also the more concrete types of ordinary and scientific discourse. Third, the theory of arbitrary objects is the only intuitive theory that I know of to yield an *arbitrarily* large number of indiscernible objects. This possibility is of obvious philosophical interest and can be used to provide a justification for Harvey Friedman's 'complete theory of everything'[viii]. Fourth, the present theory may provide a rigorous basis for understanding certain ancient views on 'one' and number. I hope to write elsewhere on some of these other topics.

§1 Objections to Cantorian Abstraction

In attempting to reconstruct Cantor's account of cardinal number, I shall make a certain simplification. Cantor supposes that sets are given with a certain order on their elements. He therefore takes the cardinal number M (double bar) of an aggregate M to be the result of a *double* abstraction on M (hence the double bar). One first abstracts from the nature of the elements in M to obtain its order type M (bar), and then abstracts from the order in M (bar) to obtain the cardinal M (double bar). I shall suppose, on the other hand, that we start off with an *unordered* set M and obtain the cardinal number of M as the result of a *single* act of abstraction on its elements.

It might be thought that, by proceeding in this way, we deprive ourselves of a significant advantage of Cantor's own account. For if the cardinal number M (double bar) of M is derived from an order type M (single bar), then the position of the units in M (single bar) may give us a clue to their identity and hence to the identity of the units in M (double bar). Now this point might indeed have some merit if the given orderings were well-orderings or if, in some other way, they assigned a unique position to each of their elements; for we might then identify the unit in terms of its position within the ordering. But the orderings, for Cantor, comprise all linear orderings - whether well-ordered or not; and so if M were taken to be the set of rationals under their natural ordering, then it is hard to see how M could be of any more help in fixing the identity of the units than the corresponding unordered set of rationals.

I am therefore inclined to think that if Cantor's account can be made to work, then so can an account that takes unordered sets as its starting point. Moreover, the alternative account is superior to Cantor's on at least two counts. First, given the notion of cardinal number for an unordered set, we can define the cardinal of an ordered set as the cardinal of the corresponding unordered set (without having to go through the order type). On the other hand, given the notion of cardinal number for an ordered set, one can only define the cardinal of an unordered set (such as a cardinal itself, on Cantor's view) if one assumes that there is a corresponding ordered set. But that every set can be ordered is a contentious assumption which it would be better to do without. Second, unordered and ordered sets are special kinds of structures. One would like to have a general acccount of structure type and it is hard to see how such an account could succeed unless it *directly* yielded the type of each kind of structure.

There are several significant objections that can be made to Cantor's account, whether stated in terms of single or double abstraction. Some of them are foreshadowed in Frege's *Grundlagen*, but I have not attempted to determine the exact relationship between my own objections and his. The first, and most glaring to contemporary philosophical sensibility, is that the account is stated in psychological terms. The units are portrayed as the product of a psychological act of abstraction; *we* abstract from the elements of a set; and the resulting number has some kind of existence in our mind 'as an intellectual image or projection' of the set. The notion of abstraction would therefore appear to be a psychological notion. But if we are to have a definition of number, then it should be stated in mathematical, or logico-mathematical, terms and certainly not in psychological terms.

Cantor himself had previously acknowledged the need to keep psychological elements out of the concepts and propositions of arithmetic. In reviewing Frege's Grundlagen[ix], he writes:

> It should also be recognized that the author has adopted the correct position in so far as he has demanded that spatial as well as temporal intuition and likewise all psychological elements must be kept out of the concepts and basic propositions of arithmetic. Only in

this was can rigorous logical purity be achieved, and thereby the justification for the application of arithmetic to imaginable objects of cognition.

So perhaps Cantor was not attempting to provide a *definition* of number in the passage quoted earlier. Perhaps for him the notion 'x is the cardinal number of y' is primitive and what we have is not a definition of the notion but a psychological account of how it is to be grasped.

However, this interpretation does not seem to square with Cantor's own understanding of the passage. For he treats it as providing a "definition of power" and appeals to it as such in his proof that equinumerous sets are of the same cardinality[x]. This therefore suggests that the psychological detail was incidental to his purposes; and, indeed, in the parallel definition of order type in part two of the *Beitrage*, the psychological element is barely present.

Whatever Cantor's view might have been, the value of his proposal will rest on whether a properly mathematical notion of abstraction can be made out and whether number can be defined, or otherwise explained, in terms of such a notion. Unfortunately, Cantor's own formulations are not of much help in this regard. For once they are stripped of their psychological trappings, nothing much seems to remain; the apparent substance of the account appears to evaporate in mental mist.

Indeed, it may be argued that no notion of Cantorian abstraction can properly meet the demands imposed upon it and that any attempt to explain number of order type in terms of abstraction must therefore be in vain. Let us consider two objections of this sort, one focussing on the notion of abstraction itself and the other on the correlative notion of 'unit'.

The first objection amounts to a proof that there can only be two numbers under the Cantorian account - 0 and 1. For let _(x) be the result of abstracting on the object x. The number of any set {x_1, x_2, ... } is then {_(x_1), _(x_2), ... }. Consider now any two objects, say Socrates and Plato. Then the number of {Socrates} is {_(Socrates)} and the number of {Plato} is {_(Plato)} and, since these two numbers are the same, _(Socrates) = _(Plato). But then the number of {Socrates, Plato} is {_(Socrates), _(Plato)}, which is the same as the number of {Socrates}! As Frege saw so clearly in the Grundlagen, §45, the account of number in terms of units seems to force us 'to ascribe to units two contradictory qualities, namely identity and distinguishability.

One might attempt to overcome this problem by making the abstraction function relative to the cardinality of the set in question. Thus how one abstracts will depend upon how many objects one is abstracting from. But this too will not do. For let N be the set of nonnegative integers and P be the set of positive integers. Then the cardinality **c** of the two sets is the same and yet the number {$\_{}_{\mathbf{c}}(1)$, $\_{}_{\mathbf{c}}(2)$, ... } of P will be a proper subset of the number {$\_{}_{\mathbf{c}}(0)$, $\_{}_{\mathbf{c}}(1)$, ... } of N.

It will not even do to make the abstraction function relative to the *set* in question. For consider the doubleton X = {Socrates, Plato}. Its cardinal number is the set of units {$\boldsymbol{u}_0$, $\boldsymbol{u}_1$}. But on what basis do we set $\_{}_{X}$(Socrates) identical to $\boldsymbol{u}_0$ rather than $\boldsymbol{u}_1$ and, given that the cardinal number of Y = {Caesar, Brutus} is also {$\boldsymbol{u}_0$, $\boldsymbol{u}_1$}, then on what basis do we set $\_{}_{Y}$(Caesar) identical to $\_{}_{X}$(Socrates) rather than to $\_{}_{X}$(Plato)?

The argument may be stated in more positive fashion. If we have properly abstracted from the members of a set, then the resulting units should be indistinguishable from one another - for abstraction should eliminate any distinctions among the objects to be abstracted beyond their merely being distinct. If, for example, the units $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ are the result of abstracting on the objects x_0 and x_1, then no distinction between x_0 and x_1, beyond their merely being distinct, should persist as a difference between $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$. Cantor himself seems to endorse such a view when he writes[xi]:

> the elements [of the cardinal number] as so-called ones have in a certain way grown together organically into one another to a *uniform whole* [my italics] in such a way that none of them has a privileged relation of rank to the others.

However, abstraction, even when it is made relative to the set upon which the abstraction is performed, violates this requirement of indistinguishability. For consider the number {_X(Socrates), _X(Plato)} that results from abstracting on the members of the set X = {Socrates, Plato}. If _X(Plato) were distinct from _X(Socrates), then it would be distinguished from _X(Socrates) by virtue of the fact that it is was the result of abstracting on Plato rather than on Socrates. Thus indistinguishability without identity appears to be impossible.

To remedy this further difficulty, one might think that abstraction should be taken to apply to the elements of a set as a whole. Thus rather than regarding the number of a set $\{x_1, x_2, ...\}$ as the result of the *multiple* applications of the abstraction function to the individual elements $x_1, x_2, ...$, we now regard the number as the result of a *single* application of the function to all of the elements $x_1, x_2, ...$ at once. There is even some evidence that Cantor himself may have favoured such an approach, for in reference to Leibniz's comments on number he writes:

> The addition of ones, however, can never serve for a definition of a number, since here the specification of the main thing, namely *how often* the ones must be added, cannot be achieved without using the number itself. This proves that the number is to be explained only as an organic unity of ones achieved by a *single act of abstraction* [my italics][xii].

But how are we to construe this single act of abstraction? Presumably, in order to give general expression to the idea that certain objects are the abstractions of certain others, taken as a whole, we must take the *set* of the former to stand in the appropriate relation to the *set* of the latter. But this relation between sets *is* just that of a cardinal number to the sets that are of that cardinal number. It is therefore no longer clear how the one notion might be used to explain the other.

The second objection I wish to consider turns on the internal constitution of the numbers themselves rather than on the external relationship of the units to other objects. The Cantorian number 1 is a set $\{\boldsymbol{u}\}$ consisting of a single unit $\boldsymbol{u}$. The Cantorian number 2 is a set $\{\boldsymbol{v}, \boldsymbol{w}\}$ consisting of two units $\boldsymbol{v}$ and $\boldsymbol{w}$. Since the units $\boldsymbol{v}$ and $\boldsymbol{w}$ are two in number, at least one of them, say $\boldsymbol{v}$, is not the unit $\boldsymbol{u}$. But why should 1 consist of the unit $\boldsymbol{u}$ rather than the unit $\boldsymbol{v}$? If we abstract on $\{\boldsymbol{u}\}$, we obtain $\{\boldsymbol{u}\}$. But if we abstract on $\{\boldsymbol{v}\}$, we obtain $\{\boldsymbol{u}\}$. Why should this be? Why should certain units be especially suited to compose a number or to be the products of abstraction and not others?

One might think that there was an easy answer to this question. For we may suppose that the units are generated in turn in much the same way as the natural numbers. Thus there will be a first unit $\boldsymbol{u}_1$, its successor $\boldsymbol{u}_2$, *its* successor $\boldsymbol{u}_3$, and so on. This then makes it natural to suppose that when we abstract from objects we go through the units in turn. The resulting natural numbers will therefore consist of sequences of units: 1 will be $\{\boldsymbol{u}_1\}$; 2 will be $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$; and so on. We thereby obtain a von Neumann style version of the Cantorian theory. But there is no need to identify $\boldsymbol{u}_n$ with the number n-1 as on von Neumann's own account. We could instead imagine that $\boldsymbol{u}_1$ was an 'original' unit, $\boldsymbol{u}_2$ a 'copy' of $\boldsymbol{u}_1$, $\boldsymbol{u}_3$ a copy of that copy, and so on.

There is, however, a general objection to any abstractionist view which would make one nonzero number a subset of another or which would even allow two numbers to overlap. For recall that the units within any given number should be indistinguishable from one another.

Suppose now that one nonzero number was included in or overlapped with another. Then the numbers would contain a common unit ***u*** and one of the numbers would contain a unit ***v*** not contained in the other. The units ***u*** and ***v*** of that number would then be distinguished by virtue of the fact that ***u*** belonged to the other number while ***v*** did not.

We face an additional problem in connection with order types, for which the generation of the units provides no help. For we need to explain not only why it is these units - rather than others - that constitute the type, we also need to explain why they are ordered in the way they are. Consider the order type of the rationals, for example. Why should the type be one in which the units enter into the ordering in the way they do rather than in the reverse ordering or in any of the other possible rearrangements of the given ordering?

This problem even arises for the finite ordinals (i.e. the order types of finite well-orderings). Consider the finite ordinal 2, for example. It contains two units, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. But why should we take one to be first and the other second, rather than the other way round? Given that one of the units belongs to the ordinal 1, it is natural to take this unit to be first. But this is merely the result of a convention; we read the ordering relation as 'less-than'. If we read it as a 'greater-than' and treated the ordinals as growing 'backwards', then it would be just as natural to take the unit from the ordinal 1 to come last rather first.

But even if this difficulty over the choice of the ordering could somehow be resolved, a difficulty would remain. For if each of the finite ordinal numbers 0, 1, 2, ... is to be included in its successor and also in _ (the order type of the natural numbers under their natural ordering), then presumably each should also be included in $\_^*$ (the order type of the nonpositive integers under their natural ordering). For how can the 'direction' of the relation make a difference to whether a relation of inclusion holds? But it can now be shown[xiii] that no matter how we choose the units or order them within a type, it is impossible for the finite ordinals to be included, in the required way, both in their finite successors and also in _ and $\_^*$. This shows how special a von Neumann style of representation is to the case of the ordinals, for it breaks down once $\_^*$ is thrown into the picture.

If we ask the general question, 'when, if at all, should one type be included in or overlap with another?', then the only possible answer would appear to be: when it or a suitable part of it can be suitably embedded in the other. But this answer cannot hold in the general case since it cannot hold in the special case above of the finite ordinals with _ and $\_^*$. Thus the only reasonably systematic account of abstraction along Cantorian lines must be one that denies that inclusion or overlap of types is ever possible.

We see from the above discussion that any satisfactory account of number along Cantorian lines must solve certain problems. Most important of all, it must be shown to rest upon a cogent account of abstraction - one that is stated in mathematical (or logico-mathematical) terms, that does not already presuppose the concept of number, and that provides us with a clear and definite understanding of what abstraction is.

But there are some more particular difficulties. First, we must explain how a number can result from abstracting on the elements of a set even though no units can be uniquely associated with each element of the set. Second, we must explain how the units within a number are indistinguishable from one another even though the units from different numbers are not. Third, we must explain why only certain units belong together in a number or order type and why the units within an order type are arranged in the way they are.

Given the seemingly intractable nature of these difficulties, it is not surprising that logicians and philosophers have despaired of developing a satisfactory account of number along these lines.

§2 The theory of variable objects

______I believe that these difficulties may be solved by appealing to the theory of variable (or arbitrary) objects developed in my 'Reasoning with Arbitrary Objects'. For the units appearing in numbers and order types may be taken to be variable objects of a suitably special kind. I shall here give an exposition of the relevant portion of the theory, though it is worth noting that the theory was originally developed with completely different applications in mind.

It is natural nowadays to think of variables as a certain kind of sign. However, I wish to think of them as a certain kind of object. Like variable signs, these objects will take other objects as values. But unlike variable signs, they will not be linguistic in character; they will not be conventional symbols for their values, but abstract objects which assume those values by way of their intrinsic nature.

This difference in the ontological status of variables will be critical for our purposes. For once it is allowed that variables are objects, it will be natural to suppose that some of them are mathematical objects and hence capable of playing the role of units in a mathematical account of number. If variables were signs, they would be entirely unsuited for this purpose. We would have a form of 'linguisticism' that was just as bad as the psychologism from which we had been trying to escape.

The variable signs of logic take values independently of one another; what value one variable sign takes is not constrained by the values that other variable signs take. However, I wish to allow for the possibility that the value of one variable object may depend upon the value taken by others. I will allow, for example, that there are variables $\boldsymbol{x}$ and $\boldsymbol{y}$ whose values are any real numbers x and y for which $x = -y$. (I use bold face italic for variables objects, light face italic for their values, and plain lower case for variable signs.)

There are two different ways in which we may think of one variable as depending upon the other. On the one hand, we can think of the dependence as two-way, with the values of the variables being simultaneously constrained. On other hand, we can think of it as one-way, with the values of one variable being given in terms of the value of the other. Thus we may either think of the variable $\boldsymbol{x}$ and $\boldsymbol{y}$ above as being simultaneously constrained by the requirement that the value of one should be the negative of the value of the others; or we may think of the value of one of the variables, say $\boldsymbol{y}$, as being given as the negative of the value of the other.

A simpler account of cardinal numbers and order types can be obtained by requiring dependence to be two-way but a somewhat more satisfactory account is obtained by allowing dependence to be one-way as well. In what follows, I shall concentrate on the two-way case but also indicate how the account should proceed in the one-way case.

A *system* of variables is a set of variables with the property that (i) any two variables in the set are co-dependent and (ii) no variable in the set depends upon any variable not in the set. A set of variables satisfying condition (ii) might be said to be *independent*. A system will then be an independent set of variables that contains no proper independent subset.

There is a natural criterion of identity for systems of variables. Let S and S' be any two systems of variables; and let us say that S and S' are *similar* under a one-one map $\boldsymbol{x}$ _ $\boldsymbol{x'}$ from the variables of S onto the variables of S' if the values taken by the variables of S are the same as the values taken by the corresponding variables of S'. Thus if S consists of the variables $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ... and S' of $\boldsymbol{x}_1'$, $\boldsymbol{x}_2'$, ..., then the systems will be similar if, for any objects a_1, a_2, ..., a_1, a_2, ... will be the respective values of $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ... just in case they are the respective values of $\boldsymbol{x}_1'$, $\boldsymbol{x}_2'$, ... The criterion of identity then states that the variables of similar systems are the same. The identity of a system of variables is given by the values that the variables assume.

Consider the special case in which S contains the single variable ***x*** and S' the single variable ***x***'. Thus ***x*** and ***x***' will be independent variables, they will depend upon nothing but themselves. The criterion then says that the variables ***x*** and ***x′*** are the same as long their values are the same.

Consider now the special case in which S contains the two variables ***x*** and ***y*** while S' contains the two variables ***x***' and ***y***'. The criterion then says that the variables {***x***, ***y***} will be the same as the variables {***x***', ***y***'} as long as the objects *a*, *b* are values of ***x***, ***y*** just in case they are values of ***x***', ***y***'.

The criterion does not state that, in this case, ***x*** = ***x***' and ***y*** = ***y***'. Indeed, consider the case above in which the value of ***y*** is the negative of the value of ***x***. Then *a*, *b* will be a value of ***x***, ***y*** just in case it is a value of ***y***, ***x***. But we do not want to say that ***x*** = ***y***.

In the case that one-way dependence is allowed the criterion for the identity of two systems of variables S and S' is somewhat more complicated. In the first place, we should require that the variables in S and S' unilaterally depend upon the same variables. And in the second place, we should require that, under some one-one map between the variables of S and S', the systems should be similar relative to any given values of the variables upon which they depend. Thus if ***x*** and ***y*** each unilaterally depend upon a single variable, then they will be the same as long as that variable is the same and, for each value of that variable, their values are the same.

We should note that the above criteria are false for variable signs. Two independent variable signs, for example, may take the same values and yet be distinct. But even though the signs are distinct, we take the corresponding variable objects to be the same. This is another respect in which it is critical to our purposes that variables should be objects rather than signs.

I have so far talked about the abstract theory of variable objects without reference to a language by which they might be defined or described. There are two linguistic aspects of the theory that will be important in what follows.

First, there is a standard way in which systems of variable objects may be defined by means of conditions on their values. These conditions will contain uninterpreted variable signs to which the definition then assigns appropriate variable objects. Consider, as an example, the definition:

LET x and y be such that: x and y are reals and y = - x.

(The 'LET' is capitalized to distinguish this generic definition from a classical definition of the usual sort). The embedded condition will define a system of two co-dependent variable objects ***x*** and ***y*** whose values are all reals *a* and *b* for which *b* = -*a*. The values assumed by ***x*** and ***y*** are exactly those that satisfy the condition when assigned to x and y. It follows from the identity criterion above that the system of variable objects determined by a definition will be unique. However, the *assignment* of variable objects to variable signs need not be unique. In the above definition, for example, ***x*** and ***y*** or ***y*** and ***x*** might equally well be assigned to x and y.

The corresponding definition of a dependent (but not co-dependent) negative variable is given by following two clauses:

LET x be such that: x is real;

LET y be such that: y = -x.

Here the first clause defines the independent real, while the second clause defines the dependent real whose values is always the negative of the value of the first real. Thus in this case, each clause picks out a unique variable object.

The second linguistic aspect concerns the use of variable signs to which a variable object has already been assigned. There are two ways to interpret sentences containing such signs. The first, which I call *representational* or *generic*, corresponds to the usual interpretation of variable signs. Whether the sentence is true will depend upon what values are assigned to the variable objects (and hence indirectly to the signs themselves). In particular, we may follow the convention that the assertion of a sentence under a generic interpretation is to be true if it is true for all values of its variables.

The second, which I call *classical* or *literal*, depends upon taking seriously the role of variable objects as intermediaries between the variable sign and its values. Under this interpretation, the sentence is taken to be true if it says something true of the variable objects signified by its variable signs. Let us suppose that the variable sign x signifies the variable object ***x***. Then under a generic reading, the assertion:

x is a variable object

will be true only if each value of ***x*** is itself a variable object. But under the literal reading, the assertion will be true, regardless of what values are assumed by ***x***.

§3 Units as Variables

I shall show how one can develop a satisfactory Cantorian account of number by construing units as variables. I here present a simplified account, which will be modified in the light of the more general considerations of the following section.

The number 1 for Cantor is of the form $\{\boldsymbol{u}^1_1\}$, the number 2 of the form $\{\boldsymbol{u}^2_1, \boldsymbol{u}^2_2\}$, the finite cardinal n of the form $\{\boldsymbol{u}^n_1, \boldsymbol{u}^n_2, ..., \boldsymbol{u}^n_n\}$, and similarly for the transfinite cardinals. The problem of defining a given cardinal is the problem of saying what its units are.

Consider first the cardinal 1. It has a single member $\boldsymbol{u}^1_1$, something that we might call 'The One'. But what is The One? Intuitively, it must be something common to all objects whatever. This therefore suggests, within the framework of variable objects, that it should be a variable object whose range is unrestricted; any object whatever can be one of its values.

But this is hardly enough to single out The One, for there will be many variables whose ranges are unrestricted. Consider, for example, two variable objects which can take any two objects as values as long as those values are distinct. Then these two variables will both have unrestricted range. We might call the variable ***u*** *a unit* or a *one* if its range is unrestricted. The problem then is to say which of the ones is *The* One.

This problem may be solved by requiring The One to be an independent variable, one not depending upon any other variables. By the Identity Criterion for systems of codependent variables, there will be at most one unrestricted and independent variable; and so the uniqueness of The One will be assured.

The additional requirement on The One can be justified in terms of the underlying operation of abstraction. The unit ***u*** in the number 1 = {***u***} can be obtained by abstracting on the sole member *a* of any singleton set {*a*}. We are therefore, in a sense, abstracting from an object that stands alone - it is, in the context of the singleton, the only object under consideration. What then corresponds to this feature of the object is that the corresponding variable ***u*** should be independent.

Consider now the number 2 = {$\boldsymbol{u}^2_1$, $\boldsymbol{u}^2_2$}. It has two members $\boldsymbol{u}^2_1$, $\boldsymbol{u}^2_2$, which we may call 'The One' and 'The Other'. Of course, The One in the number 2 is to be distinguished from the one in the number 1. It might therefore more accurately be called 'The One as opposed to The Other'. What then are The One and The Other? Again, we should take each, as something common to all objects, to be a unit or variable with unrestricted range. And again, since we only abstract on two objects, we should take the corresponding variables to be independent of all other variables. But again, these two requirements hardly suffice to determine the identity of a pair of variables $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. For given any particular object *a*, we may take the values of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ to be subject only to the condition that one of them must be *a*. Each choice of *a* will then determine different pairs of variables that satisfy our two requirements.

To fix the identity of The One and The Other, we should further require that they be *differentiated* in the sense that the value of The One should always be distinct from the value of The Other. Thus the objects u_1 and u_2 will be respective values of The One and The Other just in case they are distinct. The units of the cardinal 2 will therefore be an independent pair of differentiated units; and from the Identity Criterion it will follow that there is at most one such pair, for any one-one map between two such pairs {$\boldsymbol{v}_1$, $\boldsymbol{v}_2$} and {$\boldsymbol{w}_1$, $\boldsymbol{w}_2$} will yield a similarity.

This further requirement can again be justified in terms of the underlying operation of abstraction. For in abstracting from the elements of a doubleton in order to obtain the number 2, we do not wish to abstract away from all features of the two objects. We wish merely to take account of the fact that the two objects are distinct; and so it this feature, and this feature alone, that should be preserved under abstraction.

We already see from the case of 1 and 2 the critical role played by the notion of dependence. For it is in terms of this notion that the unit of 1 and the units of 2 are to be distinguished; for the unit of 1 does not depend upon anything else, while the units of 2 depend upon one another. Without appeal to the notion of dependence, it is hard to see how we could both allow for the presence of two units in 2 and yet distinguish them from the unit in 1.

With the numbers 1 and 2 in hand, it is straightforward to extend our account to all numbers whatever. A number can simply be defined to be a system of codependent variables constrained only by the condition that their values should be distinct from one another.

For each of the finite numbers 0, 1, 2, ..., it is possible to give especially simple definitions, without explicitly quantifying over variable objects themselves. 0 is, of course, the null set. Consider the definitions:

(1a) LET u^1_1 be such that $u^1_1 = u^1_1$;

(1b) Let 1 be such that: 1 = {u^1_1}.

The first clause defines the unit of 1 and the second clause (which is to be read classically) defines 1. Now consider the definitions:

(2a) LET u^2_1 and u^2_2 be such that u^2_1 _ u^2_2;

(2b) Let 2 be such that: 2 = {u^2_1, u^2_2}.

The first clause defines the units of 2 and the second clause defines 2. For any natural number n, the units $\boldsymbol{u}^n_1$, $\boldsymbol{u}^n_2$, ..., $\boldsymbol{u}^n_n$ of n and the number n itself can be defined in a similar way.

We may see from our account how the various problems that stood in the way of developing a satisfactory Cantorian theory can be solved. First, it is clear how the units within any number will be indistinguishable from one another. For the system that is the number will be similar to itself under any permutation of the units. Indeed, even if we fix the identity of all other objects in the universe, there is no way in which two units from a number system might be distinguished (except, of course, in terms of being distinct from one another). It is perhaps because of this absolute form of indistinguishability of the units within each cardinal number that there is such a strong temptation to suppose that they are all the same.

Second, it is clear how the units from different numbers will be distinguishable from one another and hence distinct; for they will be co-dependent with a different number of other units. To use phraseology suggested to me by David Kaplan, the unit of 1 will be one of one, the units of two will each be one of two, and so on for all the other numbers.

To resolve the difficulties over abstraction, we must first get clearer on what abstraction is. Suppose that we have various objects a_1, a_2, ... and various features of those objects (in the form of properties and relations) which we wish to preserve under abstraction. We might represent these features by a single compendious relation R. Thus the objects b_1, b_2, ... having the relevant features will simply consist in their standing in the relation R.

We may then say that the variables $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ... result from abstracting on the objects a_1, a_2, ... with respect to the relation R if, first, a_1, a_2, ... stand in the relation R and, second, the variables $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ... constitute a system constrained only by the requirement that their respective values should stand in the relation R. Of course, whether the variables satisfy the second condition will be independent of whether the objects satisfy the first condition. But, in general, it will only be because we already know that there are objects with the desired feature that we can be assured that there exists an abstraction with respect to that feature.

In the special case of numbers, the units $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ... which comprise a given number will be obtained by abstracting on the members a_1, a_2, ... of a set with respect to the relation of diversity. This is the relation that holds of the objects b_1, b_2, ... just in case any two of them (in distinct positions) are distinct. Let us note that the abstraction will not result in a unique association of units with objects, since if $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ... is the result of abstracting on a_1, a_2, ... wrt diversity then so is any permutation of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ...

Given this relational notion of abstraction, we may understand the *operation* of abstraction to be either singular or simultaneous. Suppose that the respect R with which we are to abstract is given. We may take abstraction to be an operation which, in application to each of the individual objects a_1, a_2, ... as arguments, yields variables $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ... that are the result of abstracting on a_1, a_2, ... with respect to R. Or we may take it to be an operation which, in application to a *set* of objects $\{a_1, a_2, ...\}$, yields a set of variables $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ... that are the result of abstracting on a_1, a_2, ... with respect to R.

Our worry from section 1, in regard to singular abstraction, was that it required us to make invidious distinctions among units. In virtue of what is the result _(Socrates) of abstracting on Socrates, say, $\boldsymbol{u}^2{}_1$ rather than $\boldsymbol{u}^2{}_2$? We now see that there is not a single operation of singular abstraction but as many operations as there are permutations of the individuals. Thus in addition to the operation _ there will also be an operation ß that takes Socrates into $\boldsymbol{u}^2{}_2$ and Plato into $\boldsymbol{u}^2{}_1$; and there will be no way to distinguish between _ and ß except in terms of Socrates and Plato themselves.

Our worry from section 2, in regard to simultaneous abstraction, was that it would provide us with no way to understand abstraction independently of number. But our present definitions show this worry to be unfounded. For number abstraction can be taken to be a special case of abstraction in general, which can then be defined without recourse to the concept of number.

We turn finally to the question of cogency. The definitions rest, of course, upon notions from the theory of variable objects -the notions of variable, value, and dependence. These notions may not be as familiar or as established as the notions of set or membership. But my view, which I shall not defend here, is that they are already enshrined in ordinary mathematical practice and that we are capable of achieving as clear and definite an understanding of them as of the standard notions of set theory. Moreover, they surely are mathematical or logico-mathematical notions; and since the only other notions of which we made any use were from set theory and logic, our account will provide us with an explicit mathematical definition of number.

The definitions confirm not only that number is a *notion* of mathematics but also that numbers are *objects* of mathematics. For it seems plausible to suppose that a variable will be an object of mathematics if it can (in principle) be defined entirely in mathematical terms and that a set will be an object of mathematics if its members are. But since the units can be defined entirely in mathematical terms, the numbers which are composed of them, will be mathematical objects.

§4 Order Type

I consider the extension of our account to abstraction on orderings. A (simply) ordered aggregate for Cantor is what we would now call a strict linear ordering, one whose ordering relation is asymmetric, connected and transitive. An ordered aggregate may be identified with an ordered pair <A, R>, where A is a set and R is a relation on A. We shall concentrate on the question of defining the type of the relation R; and, to this end, we shall identify R with a subset of ordered pairs from A _ A. However, nothing will turn on the identification; and, once the definition is given, it will be clear how it may be extended to orderings and to other kinds of structure.

Consider, as an example, a strict linear relation with two elements. This will be of the form {<*a*, *b*>} for *a* distinct from *b*. Its type, which we may denote by $\underline{2}$, will also be of the form {<***u***, ***v***>}, where ***u*** and ***v*** are now appropriate units obtained by abstraction from the elements *a* and *b* of a relation {<*a*, *b*>}.

But what are the units ***u*** and ***v*** (which we might designate 'The First' and 'The Second')? We cannot take them to be the units $\boldsymbol{u}^2_1$ and $\boldsymbol{u}^2_2$ of the cardinal number 2. For why should the relation type be $\{\langle \boldsymbol{u}^2_1, \boldsymbol{u}^2_2\rangle\}$ rather than $\{\langle \boldsymbol{u}^2_2, \boldsymbol{u}^2_1\rangle\}$? Indeed, if the relation type $\underline{2}$ were, say, $\{\langle \boldsymbol{u}^2_1, \boldsymbol{u}^2_2\rangle\}$, then $\boldsymbol{u}^2_1$ would be distinguished from $\boldsymbol{u}^2_2$ by virtue of the fact that it was the 'first' unit in $\underline{2}$.

One might, at this point, appeal to our previous idea that the units are generated in succession. We no longer face the previous difficulty over indistinguishability, since whereas the units within the *cardinal* number 2 are not distinguishable the units within the *ordinal* 2 should be distinguishable. We may therefore take the first unit to be generated to be first in the ordering and the second to be second.

However, the difficulty will still arise for the units within the relation type of the natural ordering for the rationals. For there is no basis upon which we can regard any unit within this ordering as first as opposed to any other. And a similar difficulty would also arise for certain types of finite relation. Consider, for example, a two-element cycle {<*a*, *b*>, <*b*, *a*>}. The units within its type should be indistinguishable, as with the finite cardinals.

Given that some of the units within a relation type are distinguishable from one another and given that the units all belong to the same system, there must be some other variable or variables within the system to which they belong. For otherwise we would just have a system of differentiated units, i.e. of units from a cardinal number, all of which would be indistinguishable from one another. And given that there are other variables in the system, it must be by reference to them that the units capable of being distinguished are indeed distinguished.

But what might this other variable or these other variables be? An important clue is provided by the intuition that the units within a relation type are not "bare" units, merely indicating one object as opposed to another, but "positional" units, which also indicate the position of an object within the relation. This suggests that the variables within a relation type should be taken to depend, not merely upon one another, but upon a relation variable whose values serve to indicate the position of the units. So in the case of the relation type $\underline{2}$, we will have a system of *three* variables - two units $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and a positional variable $\boldsymbol{R}$ - whose respective values v_1, v_2 and R are subject to the constraint that v_1 should be distinct from v_2 and R should be identical to $\{<v_1, v_2>\}$. Or again, in the case of the order type of the rationals, we will have a system consisting of a variable $\boldsymbol{w}_p$ for each rational p and a relation variable $\boldsymbol{R}$, whose respective values - w_p, for each rational p, and R - should be subject to the constraint that w_r =\= w_s for r =\= s and $\boldsymbol{R} = \{<w_r, w_s>: r < s\}$.

There are two views one might take on the relation of dependence between the units and the relation variable. On the one hand, one might suppose that the relation variable is an independent variable, whose values are all of the relations of the given type, and that the units are unilaterally dependent upon the relation variable. On the other hand, one might suppose that the relation variable and the units are codependent upon one another. Thus in the first case, the values of the units are given in terms of the value of the relation variable while, in the second case, the values of the relation variable and of the units are given together.

It still remains to define the relation type $\boldsymbol{R}^+$. It is not clear that we can take this to be the relation variable $\boldsymbol{R}$. For although $\boldsymbol{R}$ is generically of the right form, it is not clear that it is literally of the right form, i.e. a set of ordered pairs of the given type. However, we can take the relation type be the set of ordered pairs of units in which their actual position is the same as their generic position. The relation type $\underline{2}$, for example, will be $\{<\boldsymbol{v}_1, \boldsymbol{v}_2>\}$ (i.e. the relation in which The First precedes the Second), while the type of the natural ordering on the rationals will be $\{<\boldsymbol{w}_r, \boldsymbol{w}_s>: r < s\}$.

As before, by using let-clauses, we may give simple definitions of the relation types in particular cases. The definition of $\underline{2}$, for example, is given by:

(3a) LET R_2, v_1 and v_2 be such that: v_1 _ v_2 and $R_2 = \{<v_1, v_2>\}$;

(3b) Let $\underline{2}$ be such that: $\underline{2} = \{<v_1, v_2>\}$,

and, more generally, the definition of $\underline{n}$ takes the form:

(4a) LET R_n, $v_1, \ldots, v_n$ be such that: $v_1, \ldots, v_n$ are pairwise distinct and $R_n = \{<v_1, v_2>, \ldots, <v_{n-1}, v_n>\}$;

(4b) Let $\underline{n}$ be such that: $\underline{n} = \{<v_1, v_2>, \ldots, <v_{n-1}, v_n>\}$.

If the units are made to depend unilaterally upon the relation variable, the definition of $\underline{2}$ takes the following form:

(5a) LET R_2 be such that: $R_2 = \{<u, v>\}$ for some distinct u and v;

(5b) LET v_1 and v_2 be such that: $R_2 = \{<v_1, v_2>\}$;

(5c) Let $\underline{2}$ be such that: $\underline{2} = \{<v_1, v_2>\}$.

And similarly for the other cases.

The definition of the relation type $\boldsymbol{R}^+$ requires the definition of the auxiliary item $\boldsymbol{R}$, which we may call its *prototype*. The variables $\boldsymbol{R}$ and $\boldsymbol{R}^+$ are very closely related and it is important to be clear on how they are like and unlike. $\boldsymbol{R}$ is a relation *variable*; it takes all relations of a certain sort as values. $\boldsymbol{R}^+$, on the other hand, is a variable *relation*; it is a specific relation between variables of the given sort and hence itself a value of the relation variable $\boldsymbol{R}$. Although $\boldsymbol{R}$ is defined as taking certain relations as values while $\boldsymbol{R}^+$ is defined to *be* one of those values, there is a sense in which $\boldsymbol{R}^+$ itself represents the very same values as $\boldsymbol{R}$. Consider, by way of illustration, the relation type $\underline{2} = \{<\boldsymbol{v}_1, \boldsymbol{v}_2>\}$ and let $\boldsymbol{R}_2$ be the corresponding relation variable. Then for any given values v_1 and v_2 of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, $\underline{2}$ will represent the value $\{<v_1, v_2>\}$, since this is what is obtained by substituting v_1 and v_2 for $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in $\underline{2}$; and this value will be exactly the same as the value assumed by the relation variable $\boldsymbol{R}_2$ when $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ take the given values v_1 and v_2.

In case the proto-type $\boldsymbol{R}$ is defined to be *co*dependent with the units, its behaviour will closely mimic that of $\boldsymbol{R}^+$. $\boldsymbol{R}^+$ literally holds between certain units. So, for example, $\underline{2}$ relates $\boldsymbol{v}_1$ to $\boldsymbol{v}_2$ in the set-theoretic sense that the ordered pair $<\boldsymbol{v}_1, \boldsymbol{v}_2>$ is one of its members. But although $\boldsymbol{R}$ is not defined as a relation, it will hold *generically* between certain units. Thus $\boldsymbol{R}_2$ will hold between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in the sense that, for all values R_2, v_1 and v_2 of $\boldsymbol{R}_2$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, R_2 holds between v_1 and v_2. And, in general, the condition:

(*) vRw iff vR^+w,

will be true for any variables $\boldsymbol{v}$ and $\boldsymbol{w}$ codependent with $\boldsymbol{R}$, as long as the variable-signs are interpreted generically on the left and literally on the right.

The close connection between $\boldsymbol{R}$ and $\boldsymbol{R}^+$ under the codependency approach might suggest that the two should be identified. Thus the relation type $\underline{2} = \{<\boldsymbol{v}_1, \boldsymbol{v}_2>\}$ will be taken to be one and the same as the relation variable $\boldsymbol{R}$ that is codependent with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The identification of a type with its proto-type may, in its turn, be justified on the basis of a more general principle. Suppose that *S* is a system of codependent variables and that each of *x* and *y* is a variable of *S* or is a set constructed from variables of *S*. The principle then says that if *x* and *y* represent the same objects under every value assignment to the variables of the system, then *x* and *y* are the same.

I myself would not wish to endorse such a principle (I regard it with much the same suspicion as the identification of a singleton set with its member). But it is certainly possible to develop a consistent and version of variable object theory within which the principle holds[xiv].

It should be clear how the above account of relation type might be generalized. Suppose that *y* is a set $f(x_1, x_2, ...)$ built up from the distinct objects $x_1, x_2, ...$ by the usual set-theoretic means. For example: *y* might be an ordering $<A, R>$ and the objects $x_1, x_2, ...$ the elements of *A*; or *y* might be a group $<G, e, \_>$ and the objects $x_1, x_2, ...$ the elements of *G*. Any object of the same type as *y* (under the representation $f(x_1, x_2, ...)$) will be of the form $f(z_1, z_2, ...)$ for distinct objects $z_1, z_2, ...$ The units and the prototype $\boldsymbol{y}$ (under codependency) may then be defined by:

(#) LET $x_1, x_2, ...$ and y be such that (i) $x_1, x_2, ...$ are pairwise distinct and (ii) $y = f(x_1, x_2, ...)$;

and the type itself by:

(##) Let y^+ be such that $y^+ = f(x_1, x_2, ...)$.

Of course, when the objects $x_1, x_2, ...$ are infinite in number, the definitions will also be infinitary. But there is no difficulty in converting them to a finitary form or in giving a general definition of set-theoretic type.

There are various other ways in which the account might be further extended. Condition (i) in (#), that the objects x_1, x_2 be pairwise distinct, might be dropped. We would then obtain a definition of homomorphism-type rather than of isomorphism-type. We might also allow the object *y* to be built up from the constitutent objects $x_1, x_2, ...$ by other than set-theoretic means. For example, *y* might be the proposition that x_1 is identical to x_2. Its homomorphism-type would then itself be an identity proposition to the effect that one unit was identical to another.

Indeed, all that is strictly required for definitions (#) and (##) to go through is that f should be a function which is defined upon the variables $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ... as well as upon their values. But it is not then in general possible to regard $\boldsymbol{y}^+$ as a structural type, with a representative capacity that is independent of the function *f*. For this, it seems necessary that *y* should be a *complex* built up from constituents x_1, x_2, ... For such a complex there is a well-defined notion of substitution, and we may therefore take the type $\boldsymbol{y}^+$ to represent the result of substituting the values of the variables $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ... in the type for the variables themselves. The full development of this idea would require us to graft a theory of complexes onto the theory of variables sketched below. But this is not something I shall attempt.

We should note that if we apply the general account of structural type to sets of elements, we obtain something slightly different from our previous definition of number. The units in the number 2, for example, will now defined by:

LET $u^2{}_1$, $u^2{}_2$ and d be such that $u^2{}_1$ _ $u^2{}_2$ and d= {$u^2{}_1$, $u^2{}_2$}.

Thus instead of simply abstracting from two elements u_1 and u_2 with respect to difference, we abstract from the two elements u_1 and u_2 and their doubleton *d* with respect to the difference of the first two objects and their composition of the third. Since the positional variable 'd' does not serve to distinguish between the units, it can be dropped. But this is accidental to the given case; and, in general, we will need to abstract both from the units within a given structure and from the structure itself.

We should also remark that the present approach is loosely related to the linguistically oriented approaches of Nicholas White[xv] and Hartry Field[xvi]. Under one variant of White's view, which brings it closer to our own, each numeral is taken to be an open term with a parameter for an arbitrary _-progression. Thus the numeral '0' will be construed as the term 'the first member of the _-progression p', the numeral '1' as the second term of the _-progression p', and so on. Under Field's approach, the numerals are taken to be a sequence of terms whose denotations are subject only to the requirement that they should always be distinct.[xvii] Thus White's and Field's terms are linguistic analogues of our units. However, our approach has a clear advantage. For it introduces an ontology of numbers; and so there is no difficulty in making sense of quantification over numbers or in referring to sets of numbers or number functions or other such higher order entities. But under the other approaches, it is not clear how the effect of such quantification or reference is to be achieved.

§5 Comparisons

The Cantorian approach enjoys certain advantages over its rivals; and I want, in this section, to lay out what these are. I begin with the technical points in its favour and then turn to more philosophical considerations. I shall argue, in particular, that the Cantorian approach is able to make a type be an instance of itself, as with the accounts of von Neumann and Zermelo, but without making an arbitrary choice of what that instance should be.

There are some familiar technical difficulties to von Neumann's account of cardinal numbers and to other accounts of that sort. A set will have a cardinal number only if the set can be well-ordered (since the one-one correspondence between the set and an initial ordinal will induce a well-ordering on the set). The general existence of cardinals therefore depends upon the Axiom of Choice or some equivalent thereof. But it would be preferable to allow for the possibility that non-well-ordered sets have a cardinal and to be able to prove, without the benefit of the Axiom of Choice, that every set has a cardinal.

Matters are even worse when it comes to extending the von Neumann approach to order types or to structure types in general. For even with the Axiom of Choice, it is impossible to provide a suitable definition of order type. That is to say, there is no formula _(x,y) (with the intuitive meaning 'y is the order type of x') such that:

(*) _x[x is an ordering _ _y(_(x,y) & y is isomorphic to x & _x,x',y,y'[_(x,y) & _(x',y') & x isomorphic to x' _ y = y'])

is provable. Nor does it help to allow parameters z_1, ..., z_n to occur in _(x,y) and to only require that the existential generalization _z_1..._z_n(*)) of (*) be provable[xviii].

These difficulties disappear under the approaches of Russell/Frege and of Cantor. The existence of the class of sets equinumerous with a given set does not depend upon the set being well-ordered; and, likewise, nor does the existence of the corresponding set of units[xix]. Moreover, the same methods that are used to prove the existence of cardinal numbers in these cases can be used to establish the existence of order types or of structure types in general.

However, the Frege/Russell approach cannot be accommodated within ZF. Indeed, the set of all sets equinumerous with a given non-empty set can be proved not to exist (since its union will be the universal set). Cardinals should therefore be classes rather than sets, and the classes in question should submit to a theory very different from ZF. But the familiar choices here - NBG, NF, or the theory of types - are not appealing. NBG will not allow cardinals to belong to sets; NF rests upon an unintuitive notion of class; the theory of types requires an ambiguous notion of cardinal; and neither NF nor the theory of types has the power or flexibility of ZF.

The Cantorian approach, by contrast, can be grafted onto ZF without any difficulty; for the units can be treated as urelements which can then figure in sets of the cumulative hierarchy in the usual way. Somewhat surprisingly, we see from the possibility of grafting a theory of units onto ZF that the shortcomings of the Frege/Russell approach may have been exaggerated. For there is a natural connection between variables and classes; each variable may be associated with the class of its values; and each class may be associated with an independent variable whose values are the members of that class. This therefore suggests that there should be a treatment of classes in which they are capable of belonging to sets in much the same way as variables. Indeed, using the theory of variable objects as a guide, I have been able to construct a natural and powerful theory of this sort - one which contains ZF, establishes the existence of Frege-Russell cardinals and other such types, and allows classes to belong to sets with the same freedom as urelements[xx]. I shall give details elsewhere, but we should remark that it was only the conception of units as a special kind of urelement that led to the analogous conception of classes.

Let us now turn to the philosophical considerations. Following Hallett[xxi], we shall say that an account of the types of some kind is *representational* if each type of the given type is of that very type. The accounts of cardinal number given by von Neumann and Zermelo are representational in this sense: each cardinal number is of that number. Their accounts of ordinal number are not strictly representational, since the ordinal numbers will be sets rather than well-orderings, but they are readily rendered representational by equipping each of the sets with an appropriate well-ordering. On the other hand, the accounts of cardinal and ordinal number of Russell and Frege are not representational: each cardinal or ordinal number will not (as a rule) be of that number.

There is a certain technical advantage in adopting a representational account of numbers or of some other kind of type. For we can then reason directly with the types and define their basic properties and relations without making a detour through what they are types of. Under the von Neumann account of number, for example, the relation of less-than-or-equal can simply be defined as the subset relation.

There may also be a certain philosophical advantage in adopting a representational theory. For certain types do seem to be self-applicable. We do seem to conceive of the order type of a ordering, for example, as an ordering that is of that very type, though whether a cardinal number should be regarded in a similar same way as a representative item of that number is not so clear. However, along with this advantage of the accounts of von Neumann and Zermelo comes a drawback. For the accounts are in a certain sense arbitrary. They rest upon arbitrarily selecting certain representative items of a given number as being that number. If it is asked why should these representatives be taken to be the numbers rather than certain others - why von Neuman representatives, for example, rather than Zermelo representatives -then no satisfactory answer can be given.

One might attempt to mount a defence of von Neumann's account or ordinals as somehow being the simplest or most natural. But once we consider the general problem of representing orderings, it is hard to see how any such defence could succeed. For if the von Neumann ordinals, with membership as the ordering relation, provide the simplest or most natural way of representing well-orderings then, by considerations of symmetry, the von Neumann ordinals, with converse membership as the ordering relation, should provide the simplest or most natural way of representing the converses of well-ordering. But a finite well-ordering is of the same type as its converse; and so we must arbitrarily choose between membership and converse membership as the ordering relation on the corresponding von Neumann ordinal. To put the point differently: von Neumann identifies each element in a well-ordering with the set of its predecessors; but we can, with equal justice, identify each element in a converse well-ordering with the set of its successors.

The Russell\ Frege account of numbers and ordinals as equivalence classes of sets or well-orderings does not suffer from this drawback - or, at least, not to the same extent. The account deals with the representatives in an even-handed way - each is equally taken to be a member of the equivalence class. Nor does there appear to be a really compelling argument to the effect that if a number or ordinal is an equivalence classe of the items that are of that number or ordinal then it may, with equal justice, be taken to be a class of some other sort.

We see that the von Neuman/Zermelo style of account has the advantage over that of Frege/Russell of being representational while the Frege/Russell style of account has the advantage over that of von Neumann/Zermelo of being nonarbitrary. The Cantorian approach, on the other hand, combines both advantages; it provides an account of number that is both representational and non-arbitrary.

It is clear that the account is representational. But whether it is nonarbitrary is not so clear. For is it not a general defect of any representational account, and not just a particular defect of the accounts of von Neumann and Zermelo, that it will involve an arbitrary choice of representatives? For representatives of the type must be chosen; and will there not always be other representatives that might be chosen with equal justice?

The difficulty might be raised in a somewhat sharper form. Any type should be what is common, in the way characteristic of a type, to all of its instances. The difficulty then is in seeing how what is common to a given range of instances can itself be one of those instances. For it would then have a specificity as one of the instances that would appear to disqualify as being what is common to all of the instances. (We have here a form of the 'Third Man'.)

In order to meet this challenge, we cannot simply take for granted that the items chosen to be types are what is common to their instances. We must demonstrate how this is so. And this requires that we first define (or otherwise explain) what it is for one item to be the type, of the kind in question, of another item and that we then show, on the basis of the definition, how each type will, in some intuitively satisfying way, be what is common to its various instances. There is also a formal requirement that must be satisfied. For we must show, again on the basis of the definition, that each item will have a single type whose instances are all those items suitably related to the given item.

We should note that it is hard to see how one could satisfy the first two requirements, and thereby provide an intuitively satisfying account of the relation of type to instance for the types in question, without also providing a general account of the type-instance relation. Thus one test of whether we have an intuitively satisfactory account of the relation in a given case is that it should generalize to other cases.

The three requirements can be met in the case of the Frege-Russell approach to numbers. The number of a set is the class of equinumerous sets. With number-of so defined, it is clear that the number of each set is unique. Moreover, with number-of so defined, each number will, in an intuitively satisfying way, be what is common to its instances. Indeed, the sense in which it is what is common to its instances is the same as the sense in which a class is what is common to its members; the commonality of types is but a special case of the commonality of classes. We should note that this account of number-of also readily yields a general account of the type-of relation. For given an underlying equivalence relation, the type of an item will be the equivalence class containing that item as a member.

We can now better understand how the Russell/Frege account is able to withstand the charge of arbitrariness. For effectively to challenge the idea that the number of a set is the class of equinumerous sets, we must effectively challenge the idea that a class is what is common to its members; and there appears to be no plausible way in which this might be done. If one follows Benacerraf[xxii] and thinks of numbers as being defined by their natural ordering, then it does indeed appear to be arbitrary what classes we take the numbers to be, assuming we take them to be classes at all. But if it is built into our conception of number that they should be what is common to equinumerous sets (or equinumerous concepts or what have you), then the choice of numbers as classes of such items is not arbitrary and might even appear inevitable[xxiii].

The three requirements cannot, however, be met in the case of the von Neumann/Zermelo approach to numbers. For there is essentially only one way to define the number-of relation under this approach. One first gives an intrinsic account of number (thus on von Neumann's account, a cardinal number will be an initial transitive and _-connected set); and one then defines m to be the number of a set M if m *is* a number and is equinumerous with M. But with number-of so understood, a number is not, in any intuitive sense, what is common to its instances. For given an equivalence class of numbers, how can the fact that one of its members is an initial ordinal or the like provide any basis for supposing that it is, in some intuitive sense, what is common to the other members? Indeed, we have no basis, in this case, for supposing that a number is *something* common to the other members, let alone *what* is common.

Nor is this failing peculiar to the case at hand. The accounts of von Neumann and Zermelo are of the following general sort. We are given a certain equivalence relation, let us call it 'resemblance', in terms of which the types are to be defined. Certain items from the domain of the relation are picked out as privileged in terms of their intrinsic properties (and not in terms of their relationship to the items that they resemble); and an item is then defined to be the type of another if it is privileged and resembles the other. Thus the type-instance relation is defined as a restriction on resemblance.

But it is hard to see how any such account could meet our requirements. For given an equivalence class of resembling items, how can the intrinsic features of one of them provide a basis for supposing that it is something common to the others?

If we are to obtain a satisfactory representational account, some other way of defining the type-instance relation must be found. It must be supposed that there is some relation that holds between a type and its instances that is other than resemblance and upon the basis of which it can be shown that the type is common to its instances; and the type-instance relation should be so defined, or so explained, that it is then clear that this other relation does indeed hold between a type and its instances. Thus even though the type will resemble its instances, we must have an independent account of its relation to the instances, one tied to its status as a common type rather than as a resembling instance.

This is not to deny that there might also be some intrinsic way of picking out the types from among the classes of resembling items. But if this is so, then it must be true of the items picked out in this way that they stand in the required commonality-inducing relation to the items that they resemble. On a traditional conception of abstraction, for example, the colour green will merely have the intrinsic property of being green - all the other intrinsic properties of green things will have been abstracted away. But if we ask 'why should such an object be regarded as a type?', then the answer will rest on the fact that its intrinsic properties are those common to its instances.

Great as these additional demands on a representational theory may be, the Cantorian approach to types is able to satisfy them. A Cantorian type is what one might call a *form*, an abstract or non-linguistic counterpart to a scheme. Any form can be obtained by substituting variables for the constituents in some complex, just as a scheme can be obtained by substituting schematic letters for constituents in an expression. The Cantorian number 2, for example, is the result of substituting variables for the constituents *x* and *y* in a doubleton {*x*, *y*}; and the relation type of the natural ordering on the rationals is the result of substituting variables for the rationals in that ordering.

A form, in its turn, will have substitution instances, that result from replacing the constituent variables in the form by their values. Now it is evident that the form is something common to its substitution instances, just as it is evident that a class is something common to its members. And so, given that the instances of a Cantorian type or form are the same as its substitution instances, there will be a real basis, in this case, for the claim that the type is something common to its instances.

Under the Frege/Russell approach, the underlying relation between a type and its instances was that of a class to its members and there was just one object, the equivalence class, which stood in that relation to all and only the instances of the type. But in the present case, there will be many such objects - there will be many forms that are of the given type and yet have exactly the same substitution instances as the type. For example, if ***u*** and ***v*** are two units from any number greater than 1, then {<***u***, ***v***>} and {<***v***, ***u***>> will both be relations of type 2 whose substitution-instances are exactly the relations of type 2. We therefore face the problem of distinguishing that form which is the type - call it the *canonical* form - from all of the other forms and explaining why it should be regarded as *the* common element of the various instances.

When one considers the question of what the canonical form might be taken to be, there appears to be only one reasonable view. To fix our ideas, let us consider the problem of determining the canonical form of an ordered pair of distinct elements. This will itself be an ordered pair <***u***, ***v***>, where ***u*** and ***v*** are variables whose values are all those distinct objects *u* and *v* for which *u* =\= *v*. Our problem, then, is to identify the variables ***u*** and ***v***.

If the canonical form <***u***, ***v***> is to be distinguished from all of the other forms <***w***, ***x***>, then it must somehow be evident from the nature of ***u*** and ***v*** that they are to occupy the first and second position in the ordered pair <***u***, ***v***>; and this suggests that the variables ***u*** and ***v*** must themselves in some sense be the first and second members of an ordered pair. But what can this mean? Clearly, it does not mean that they are the first and second members of *some* ordered pair, since that hardly suffices to distinguish them from any two other variables. Nor does it mean that they are the first and second members of the ordered pair <***u***, ***v***>, since by same token ***v*** and ***u*** are the first and second members of the ordered pair <***v***, ***u***>. All it seems can be meant is that they are the first and second members of an *arbitrary* ordered pair. Thus it should be supposed that there is an independent variable ***p***, whose values are all ordered pairs of distinct elements, and that ***u*** and ***v*** are the dependent variables whose values are the first and second members of ***p***. (In this particular case, ***u*** and ***v*** can be defined independently of one another; but in general this will not be possible.)

One might wish for a more systematic justification for the choice of the canonical form. But still, the argument we have given seem to leave little real indeterminacy in what the choice should be. Perhaps the only real room for doubt is over the dependency status of the constituent variables. We have taken the variables in the type <***u***, ***v***> to be *dependent* upon an independent variable ***p*** whose values are all ordered pairs of distinct elements. But could we not, with equal justice, have taken these variables to be *codependent* with the variable ***p***?

There is, however, something counter-intuitive about taking the variables to be codependent (even though, for reasons of technical simplicity, we may treat them as if they are). For we think of ***u*** and ***v*** as the constituents which occupy the first and second positions in a *given* ordered pair. Thus we take what they are to be dependent upon what the given ordered pair ***p*** happens to be, and not vice versa. Moreover, this intuitive view is in line with the general policy of only assuming those dependencies required for the purposes at hand. For in order appropriately to fix the identities of ***u*** and ***v***, we only require dependence, not codependence, upon ***p***.

It should be noted that, under the present approach, we may provide an *intrinsic* characterization of the canonical forms and hence of the types. For suppose that *f* is a form F($\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ...) with variables $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ... Let ***v*** be the independent variable whose values are all substitution-instances of *f*. Then for *f* to be a *canonical* form, we require that $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ... should be the variables whose values v_1, v_2, ..., for a given value *v* of ***v***, are those for which $v = F(v_1, v_2, ...)$. But we also have, in this case, a justification for the intrinsic characterization in terms of how the type relates to its instances. For the type will be common to its instances in the manner of a form; and the constituents of the type will be common to the constituents of the instances in the way we have described.

§6 Relationship to Structuralism.

Let me conclude with some brief remarks on the bearing of the present approach on the issue of structuralism.[xxiv]

There are various ways to understand structuralism within the philosophy of mathematics, but the one most pertinent here concerns what one might call *representational* structuralism. Consider a categorical mathematical theory, such as the theory of numbers. Then from among all of the models that make the theory true, it is often supposed that there is a particular model - the intended model - which the theory is properly about. According to the representational structuralist, what distinguishes this model from all of the others is its representational role. The model somehow serves to represent all of the other models, with each element or relation of the model somehow serving to represent the elements and relations of the other models.

However, although several structuralists have been tempted by a view of this sort, they have never made clear what this representative model is meant to be. Nor have they given us any reason to believe that, from among all of the models of a categorical theory, there is indeed a unique model that is representative of the others.

It is here that our account of Cantorian abstraction may be of some help. For we may define a representational Cantorian type of a class of isomorphic models along the lines of our definition of order-type. In the case of arithmetic, for example, the number 0 will be the first element in an arbitrary _-progression, the number 1 the second element, and so on. Or, to be more exact, the prototype in this case will be an independent variable ***N*** whose values are all of the _-progressions, the numbers 0, 1, ... will be variables $\boldsymbol{n}_0$, $\boldsymbol{n}_1$, ..., dependent upon ***N***, whose values for any particular value *N of* ***N*** are the first, second, ... members of *N*, and ***n*** will be the successor of ***m*** if, for any value *N* of ***N***, its value is the successor of the value of ***m*** in *N*. Moreover, given such a definition, it may then be shown that the type is unique and itself a member of the class of models.

There are, however, definite limits on how far the present approach should be pushed. In the first place, even though the approach provides a defensible notion of a representative model, it does not commit one to the view that this is what any particular theory is properly about. Indeed, even if one believes that numbers are the results of Cantorian abstraction, one can still take them to be the result of abstraction on sets, as Cantor does, rather than on models. Thus we should distinguish between *Dedekindian* abstractionism, the view that the objects of a given mathematical theory are the objects or 'positions' in its representative model, and *Cantorian* abstractionism, the view that they are the result of abstracting on structurally similar objects.

Second, although this form of structuralism may be applied to a wide range of theories, it cannot sensibly be applied to the theory of variable objects itself. For under the intended interpretation of that theory, the domain will include various variables that are genuinely independent, whereas under the structuralist interpretation these, and all the other, variables in the domain will in reality be 'higher order' variables whose values will depend upon what model for the theory is under consideration. It follows that any doubts one might have concerning the uniqueness of the intended interpretation for variable object theory will transfer to the structuralist interpretation of other theories. Thus nothing in the present approach will serve to combat radical scepticism over whether there is a unique interpretation for any mathematical theory.

Third, the present approach will be incompatible with an austere form of structuralism, according to which the objects of the intended model possess no simple properties and enter into no simple relations beyond those given by the model itself. In the case of arithmetic, for example, the objects of the intended model will not only possess the internal property of being a number and enter into the internal relation of successor, they will also have the external property of being a variable and enter into the external relation of variable to value. Indeed, the objects of the model will be externally related by this means to every object whatever.

I myself do not find this a defect in the approach, since I doubt that there is any reasonable way in which the representational and austere forms of structuralism might be combined.

§7 Formal Addendum

I sketch how the above application of abstraction theory may be formalized. I begin with a system that only allows for one-way dependence and then consider how to accommodate two-way dependence.

The formalization will proceed within a version of ZF with individuals. It will not much matter whether we have the Axiom of Choice, though we do assume Foundation. There is one additional primitive 'Val', which is a one-place predicate. Intuitively, we think of 'Val' as being true of the basic value assignments to variable objects. We often write 'v _ Val' though, strictly, we should write 'Val(v)'. From 'Val' may be determined the variables and the relationships of dependence; for the variables will be the members of the domains of the basic value assignment, and two variables will be codependent when they belong to the domain of the same basic value assignment.

We use standard notions of set theory. In particular, we take a function to be a set of ordered pairs whose second member, for a given first member, is always unique; and we write 'Dom(f) = x' to indicate that f is a function whose domain is x. We also use the following definitions concerning variable objects:

System.

'System(S)' for 'S is the domain of some _ _ Val';

Variable Object.

'Var(v)' for 'v is in some system';

Value-assigments for S.

'_ _ Val_S' for 'Val(_) & Dom(_) = S'.

Similarity of Systems under a function f

'S $_f$ S'' for 'System(S), System(S'), f is 1-1 function from S onto S' and, for all functions _ _ $Val_{S'}$, __f _ Val __ _ _ Val'.

For the purpose of understanding the following definition, we should think of [x, _(_)] as a *quasi-system*, with the set x of objects in place of a system S of variables and with the 'value-assignments' _ on x specified by the formula _(_).

Similarity of a system to a quasi-system under f.

'S $_f$ [x, _(_)]' for 'S is a system, f is a 1-1 function from S onto x and __f _ Val __ _(_) whenever Dom(_) = x',

where _(_) a formula that does not contain S, x, or f.

Similarity of Systems.

S _ S' for 'S $_f$ S' for some f', and

'S _ [x, _(_)]' for 'S $_f$ [x, _(_)] for some f'.

The axioms for the system of variable objects (without one-way dependence) consist of those for ZF with urelements and the following additional axioms for variables:

Functionality.

_ _ Val _ _ is a function.

Atomicity.

Var(x) _ - Set(x).

Disjointness.

_, ß _ Val _ [Dom(_) = Dom(ß) v Dom(_) disjoint from Dom(ß)] Identity.

S _ S' _ S = S'.

Abstraction.

__(_(_) & _ is a function with domain x) _ _S(S _ [_(_), x]),

where _(_) is a formula that does not contain the predicate Val.

We might think of x, in the formulation of the Abstraction Axiom, as a set of argument-places {x_1, x_2, ...}. The formula _(_) then expresses a relation R whose extension consists of the tples <_($_1$), _(x_2), ... > where _ is any function with domain x that satisfies _(_). The system S of variables posited by the Axiom are then those that can be obtained by abstracting on R.

Given our notion of function, a value assignment must always be defined on every variable in its domain. But we might broaden the notion of a function so as to allow a value assignment to be undefined on some of the variables in its domain. There are other ways in which the Axiom might be strengthened. But some restriction on the form of _(_) is required; for otherwise we could prove the existence of a variable whose values are all those variables that did not take themselves as values and thereby establish a form of Russell's paradox.

We have simple definitions of cardinality. The set c will be *a cardinal* if it is a system of variables and if, for all functions _ with domain c, _ _ Val$_c$ iff _ is one-one'. The set c will then be a *cardinal of* the set x if it is a cardinal equinumerous with x.

From the Abstraction Axiom, it follows that every set x has a cardinal, for we may let _(_) be the condition that _ is a one-one function on x. And from the Identity Axiom, it then follows that the cardinal of any set is unique. Similar definitions and proofs can be given for order type.

We may also give a definition of cardinal-of in terms of representation. Let us say that the object x is *constructed from* the system S if it is either a variable of S or an urelement that is not a variable of S or a set whose transitive closure contains no variables not in S. Let us use V_S for the objects that can be constructed from S. (Of course, V_S is not a set in ZF but is given by a formula; and similarly for some of the definitions below). Now suppose that S is a system and _ a basic value assignment with domain S. Then _ may be extended to an assignment _$^+$ on V_S by means of the following rules:

(i) if x is a variable from S, then _$^+$(x) = _(x);

(ii) if x is a constant (i.e. an urelement that is not a variable), then _(x) = x;

(iii) if x is a set, then _$^+$(x) = {_$^+$(y): y _ x}.

Let us say that an x *represents* y if, for system S and some _ with domain S, x _ V_S and _$^+$(x) = y. It may then be shown that each cardinal represents exactly the sets to which it is equinumerous. In order to obtain a system that tolerates one-way dependence, we should think of the system to which a given dependent variable belongs as also containing the variables upon which it depends. The Disjointness Axiom now becomes:

_, ß _ Val _ [(Dom(_) _ Dom(ß)) v (Dom(ß) _ Dom(_)) v (Dom(_) disjoint from Dom(ß))].

In formulating the identity axiom, we require that the similarity map f between systems S and S' should be an identity on any proper subsystems. And Abstraction should be supplemented with:

[System(S) & ___Val$_S$_ß(Dom(ß) = x & ß _ _ & _(_, ß)) _ _S$^+$_f(f is an identity on x & S _$_f$ [x, ___Val$_S$(ß _ _ & _(_, ß))],

where again _(_, ß) is a formula that does not contain the predicate Val.

Consistency may be proved by the following method. We start with a standard model for set theory with urelements in which the cardinality of the urelements is the same as that of the universe as a whole. We pick out the definable classes of functions on any given set; and to each 'similarity class' of such definable classes we then assign a system of variables from among the original urelements. By using this assignment, we can define a model for the system with two-way dependence; and by appropriately iterating the procedure, we may construct a model for the system with one-way dependence[xxv].

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[i] 'Beitrage zur Begrundung der transfiniten Mengen-lehren', translated as 'Contributions to the Founding of the Theory of Transfinite Numbers' by Philip E. B. Jourdain (New York:Dover, 1955).

[ii] In the translation of W. W. Beman (La Salle:Open Court Publishing Company, 1948).

[iii] Dummett, M [1991] 'Frege: Philosophy of Mathematics', Harvard University Press: Cambridge.([91], p. 80)

[iv] P. 1 of 'Cantorian Set Theory and Limitation of Size' (Oxford:Clarendon Press, 1984).

[v] In 'Frege: Importance and Legacy', ed. M. Schirn, Berlin: Walter de Gruyter (1996), pp. 93-6.

[vi] ibid., p. 96.

[vii] Published by Oxford: Blackwell, 1985.

[viii] Friedman, H., [97] 'A Complete Theory of Everything: Validity in the Universal Domain', unpublished manuscript.

[ix] Deutsche Litteraturzeitung, no. 20 (Berlin: May 16, 1885), p. 440.

[x] Beitrage, p. 483.

[xi] 'Mitteilungen zur Lehre vom Transfiniten I, II', Zietschrift fur Philosophie under philosophische Kritik 91, 1887-8, p. 379.

[xii] Ibid, p. 381, n. 1. The passage is quoted in Hallett's book, p. 131, where further textual evidence bearing on the issue is discussed.

[xiii] The proof is as follows. The unit ***u*** of 1 appears in each of 1, 2, ... So either infinitely many units follow ***u*** or infinitely many units precede ***u*** in the subsequent finite ordinals. In the first case, the finite ordinals cannot all be included in _*; and in the second case, they cannot all be included in _.

[xiv] There is a difficulty though. Suppose we define ***u*** and ***v*** by:

LET u and v be such that: u = v.

Then the sets <***u***,***v***> and <***v***,***u***> are distinct and yet always take the same values. To avoid this difficulty, we should require under the present approach that it always be possible for the variables of a system to take distinct values.

[xv] 'What Numbers Are', Synthese 27, 1974, 111-24.

[xvi] 'Quine and the Correspondence Theory', Philosophical Review 83 (1974), 200-28.

[xvii] Analogous treatments of instantial terms in quantificational reasoning, either as open functional terms or as 'ambiguous' names, are considered in 'Reasoning with Arbitrary Objects', 136-142.

[xviii] In NBG with Global Choice we may of course give a parametrized definition of order type but not an unparametrized definition.

[xix] It therefore appears that Hallett's remark on p. 136 of his book:

> With hindsight we know that the only way to obtain a representational theory [one in which each cardinal is of that cardinal] is to appeal to some strong *ordinal* notions.

is unwarranted.

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xx Somewhat similar theories have been proposed by A. Church (in 'Set Theory with a Universal Set', *Proceedings of the Tarski Symposium*, Proceedings of Symposia in Pure Mathematics XXV, ed. L. Henkin, Providence RI (1974), 297-308), and also by U. Oswald, 'Fragmente von 'New Foundations' und Typentheorie', Ph. D. thesis, ETH Zurich (1976). But these theories are much less natural and powerful than my own. I should also mention my theory of Fregean Abstraction in 'Philosophy of Mathematics Today', ed. M. Schirn, Oxford:Clarendon Press (1998), pp. 503-630), which is based upon second-order logic rather than set theory and is much closer in spirit to Frege's original ideas.

xxi ibid, p. 136.

xxii 'What Numbers Could Not Be', Philosophical Review 74 (1965).

xxiii Hale has argued for a similar conclusion, though by very dissimilar means, in section 8.II of 'Abstract Objects', Oxford: Blackwell (1987).

xxiv Originally, this section contained a much fuller discussion but was cut at the suggestion of the editors.

xxv I should like to thank the referee for good advice, and the members of a philosophy of mathematics seminar at UCLA and of a methodology seminar at Columbia University for their many helpful comments. I am especially grateful to Tony Martin and David Kaplan.